Journal of Progressive Research in Mathematics
Vol 18,2 - 2021 — E-ISSN: 2395-0218
http://www.scitecresearch.com/journals/index.php/jprm

Development of the entire representation using the roots

Muna Tabuni

Department of Mathematics, University of Tripoli, Libya
Email: *mona_bashat@yahoo.co.uk*
Email: M.Tabuni@uot.edu.ly

Received: April 2, 2021; Accepted: April 27, 2021; Published: May 31, 2021

Abstract

One of the most important features of the analytic function is the ability to build it by knowing its zeros.
In this work we display a point by point think about of how to build the entire function using the roots with detailed mathematical proof.
The Detailed study of entire representation is given.
The roots of this entire theta function and there action are highlighted.
The number of the roots of Study function is same the dimension of the finite space and characterize it.

Keywords

Entire functions, roots, Development, build.

1. Introduction

The analytic function is very important and has been presented in many studies and has multiple uses in mathematics and physics [1, 2, 3, 15, 16, 17].
This present work sheds light on the entire function in the finite quantum spaces.
Here we center on the roots of our function and there activity.
We studied the the ability to build it by knowing the roots and we have provided extensive evidence for this.
[13] has been examined the movement of the zeros and the behavior of there curves. Ref [12, 13] has considered the analytic theta functions.
By Knowing the roots, we know the dimension of study space.
Ref [13] has been examined the properties of the roots.
Ref. [15, 14] have Created the entire function using the knowing roots.
We expect that the roots are known, and that they fulfill Eq. (9).

How to cite this paper: M.Tabuni (2021) Development of the entire representation using the roots. Journal of Progressive Research in Mathematics, 18(2), 11-19. Retrieved from http://scitecresearch.com/journals/index.php/jprm/article/view/2042

Or we assume that we have $\mathfrak{D}-1$ roots and we calculate the unknown root using Eq. (9).
We proved that the entire representation have obtained by the roots.

2 The entire theta functions

We consider the limited dimension vector space $\mathfrak{H}$ with dimension $\mathfrak{D}$.
And we define the momentum states $M_g\rangle$ as

$$|M_j\rangle = \mathfrak{F}|P_j\rangle = \mathfrak{D}^{-1/2}\sum_l (\exp\left[i\frac{2\pi j}{\mathfrak{D}}\right])|P_l\rangle, \tag{1}$$

where P_j is the position states with j is integer in $\mathfrak{D}$, and $\mathfrak{F}$ is the Fourier operator which defined as

$$\mathfrak{F} = \frac{1}{\sqrt{\mathfrak{D}}}\sum_j\sum_l (e^{(i(2\pi j)/(\mathfrak{D}))})|P_j\rangle\langle P_l|. \tag{2}$$

Consider the state

$$F = \sum_j F_j P_j; \quad , \tag{3}$$

with property:

$$\sum_j |F_j|^2 = 1; \quad , \tag{4}$$

Ref [12, 13] represented the entire representations of limited dimension vector space as following: Consider the state F of Eq.(3), with entire theta function

$$f(z) = \frac{1}{\sqrt{\sqrt{\pi}}}\sum_{j=0}^{\mathfrak{D}-1} F_j \vartheta_3[\pi j\mathfrak{D}^{-1} - z\sqrt{\frac{\pi}{2\mathfrak{D}}}; i\mathfrak{D}^{-1}] \tag{5}$$

with the periodicity property, meaning that the function after one period $\rho = \sqrt{2\pi\mathfrak{D}}$ the function repeat its value:

$$f\left[z+\rho\right] = f(z), \tag{6}$$

In addition to the property

$$f\left[z+i\rho\right] = f(z)\exp\left[\pi\mathfrak{D} - i\rho z\right] \tag{7}$$

Here ϑ_3 is the Jacobean Theta function:

$$\vartheta_3(u,\tau) = \sum_{n=-\infty}^{\infty}\exp(i\pi\tau n^2 + i2nu). \tag{8}$$

This entire function $f(z)$ is defined on a Rectangular area

With the condition :

$$\sum_{n=1}^{\mathfrak{D}} \mathfrak{z}_n = (2\pi)^{1/2}\mathfrak{D}^{3/2}(s+ih) + \left(\frac{\pi}{2}\right)^{1/2}\mathfrak{D}^{3/2}(1+i), \tag{9}$$

where s, h are integers.

2.1 Special inner product

$$2^{-1/2}\pi^{-1}\frac{1}{\pi\sqrt{2}}\mathfrak{D}^{-3/2}\quad \int_S \quad d^2z\exp\left(-z_I^2\right)\Theta_3[\pi l\mathfrak{D}^{-1}-z\sqrt{\frac{\pi}{\mathfrak{D}}}\pi^{1/2}(2\mathfrak{D})^{-1/2};i\mathfrak{D}^{-1}]$$
$$\times \quad \Theta_3[\pi j\mathfrak{D}^{-1}-z^*\sqrt{\frac{\pi}{\mathfrak{D}}};i\frac{i}{\mathfrak{D}}]=\delta(j,l), \tag{10}$$

where $\delta(j,l)$ is Dirac delta function defined as

$$\delta(j,l)=\begin{cases}0 & j\neq l\\ 1 & j=l\end{cases}, (11)$$

(12)

Follows from the ref. [15, 13]. We introduce the proof:

2.2 Proof

We start with the definition of Theta function:

$$\Theta_3(u;\tau)=\sum_{n=-\infty}^{\infty}\exp(i\pi\tau n^2+i2nu), \tag{13}$$

Put

$$\c{R}=2^{-1/2}\pi^{-1}\mathfrak{D}^{-3/2}\quad \int_S \quad d^2z\exp\left(-z_I^2\right)\Theta_3[\pi n\mathfrak{D}^{-1}-z\sqrt{\frac{\pi}{\mathfrak{D}}};i\mathfrak{D}^{-1}]$$
$$\times \quad \Theta_3[\pi m\mathfrak{D}^{-1}-z^*\sqrt{\frac{\pi}{\mathfrak{D}}};i\frac{i}{\mathfrak{D}}], \tag{14}$$

and

$$\acute{R}=\int_0^{\sqrt{(2\pi\mathfrak{D})}}dz_R e^{\left[-i\sqrt{(\frac{2\pi}{\mathfrak{D}}}(k+\ell)z_R\right]=\sqrt{2\pi\mathfrak{D}}\delta(k,-\ell)} \tag{15}$$

Hence

$$\c{R} \quad = \quad \frac{1}{\pi\sqrt{2\mathfrak{D}^3}}\sum_{k,\ell}e^{\left[\frac{i2\pi(km+\ell n)}{\mathfrak{D}}\right]}\acute{R}$$

$$\times\int_0^{\sqrt{(2\pi\mathfrak{D})}}dz_I\exp\left[-z_I^2+\sqrt{\frac{2\pi}{\mathfrak{D}}}(k-\ell)z_I-\left(\frac{\pi}{\mathfrak{D}}\right)(k^2+\ell^2)\right]; \tag{16}$$

We substitute Eq. (15) into the Eq. (16)

$$\c{R} \quad = \quad \frac{1}{\mathfrak{D}\sqrt{\pi}}\sum_{k=-\infty}^{\infty}\exp\left[\frac{i2\pi k(m-n)}{\mathfrak{D}}\right]$$
$$\times \quad \int_0^{\sqrt{(2\pi\mathfrak{D})}}dz_I\exp\left\{-\left[z_I-\sqrt{\frac{2\pi}{\mathfrak{D}}}k\right]^2\right\} \tag{17}$$

from the periodicity we get

$$\begin{aligned} \text{Ŗ} &= \frac{1}{\mathfrak{D}\sqrt{\pi}} \sum_{k=0}^{\mathfrak{D}-1} e^{\left[\frac{i2\pi k_0(m-n)}{\mathfrak{D}}\right]} \\ &\times \sum_{k=-\infty}^{\infty} \int_0^{\sqrt{2\pi\mathfrak{D}}} dz_I e^{\left\{-\left[z_I - \sqrt{\frac{2\pi}{\mathfrak{D}}}(k\mathfrak{D}+k_0)\right]^2\right\}}. \end{aligned} \tag{18}$$

Now it is easy to get

$$\begin{aligned} &\sum_{k=-\infty}^{\infty} \int_0^{\sqrt{2\pi\mathfrak{D}}} dz_I \exp\left\{-\left[z_I - \sqrt{\frac{2\pi}{\mathfrak{D}}}(k\mathfrak{D}+k_0)\right]^2\right\} \\ &= \int_{-\infty}^{\infty} dz_I e^{\left\{-\left[z_I - \sqrt{\frac{2\pi}{\mathfrak{D}}}k_0\right]^2\right\}} = \sqrt{\pi}. \end{aligned} \tag{19}$$

Now

$$\begin{aligned} \text{Ŗ} &= \frac{1}{\sqrt{\pi}} \mathfrak{D}^{-1} \sum_{k=0}^{\mathfrak{D}-1} e^{\left[\frac{i2\pi k_0(m-n)}{\mathfrak{D}}\right]} \sqrt{\pi} \\ &= \mathfrak{D}^{-1} \sum_{k=0}^{\mathfrak{D}-1} e^{\left[\frac{i2\pi k_0(m-n)}{\mathfrak{D}}\right]} \\ &= \delta(j,l) \end{aligned}$$

Ref[15, 13] has been used that to prove:

$$\begin{aligned} F_j &= \frac{1}{\sqrt{2\mathfrak{D}^3\sqrt{\pi^3}}} \\ &\times \int_S \mathfrak{D}^2 z e^{\left(-z_I^2\right)} \Theta_3\left[\frac{\pi j}{\mathfrak{D}} - z\sqrt{\frac{\pi}{\mathfrak{D}}}; \frac{i}{\mathfrak{D}}\right] f(z^*) \end{aligned} \tag{20}$$

2.3 The roots of the entire Representation

Consider an entire theta function and its roots $\mathfrak{z}_n$,
In order to study this function and know its properties, it is sufficient to study its zeros.
Also, to know the finite vector space of a study function, it suffices to know its roots and study its motion.

These roots give a Sufficient definition for the limited dimension vector space.
By knowing the number of roots of the theta function we know the dimension of the limited dimension vector space.
The movement of these roots over the time has been studied.
The sum of these roots is equal:

$$\sum_{n=1}^{\mathfrak{D}} \mathfrak{z}_n = \sqrt{2\pi\mathfrak{D}^3}(s+ih) + \sqrt{\frac{\pi}{2}\mathfrak{D}^3}(1+i).$$

3 Development of the entire representation using the roots

One of the most amazing properties of the study function is that it can be formed from its roots.
Ref. [15, 14] have created the study function from its roots. Let's say the $\mathfrak{D}$ roots $\mathfrak{Z}_n$ are known, and that they fulfill Eq. (9). Or we assume that one of these roots is unknown and we get it using the Eq. (9)

The entire theta function $f(z)$ is defined as following

$$f(z) \quad = \quad Xe^{\left[-i\sqrt{\frac{2\pi}{\mathfrak{D}}}hz\right]}\prod_{n=1}^{\mathfrak{D}}\theta\left[\sqrt{\frac{\pi}{2\mathfrak{D}}}(z-\zeta_n)+\frac{\pi(1+i)}{2};\ i\right]; \tag{21}$$

where $h \in \mathbb{Z}$ in Eq. (9), and X is a consistent fufill the eq4.

3.1 Proof

We can prove the equality of Eqs. (5) and (21) as following.

We consider the identity

$$\sum_{m=-\infty}^{\infty} x^{m^2}z^{2m} = \prod_{n=1}^{\infty}(1-x^{2n})(1+x^{2n-1}z^2)(1+x^{2n-1}z^{-2}) \tag{22}$$

The Jacobi Theta functions is defined as

$$\vartheta(z;\tau) = \sum_{m=-\infty}^{\infty} exp(i\pi\tau m^2 + i2mz), \tag{23}$$

$$= \sum_{m=-\infty}^{\infty}\left(e^{i\pi\tau}\right)^{m^2}\left(e^{iz}\right)^{2m}, \tag{24}$$

$$= \prod_{n=1}^{\infty}(1-(e^{i\pi\tau})^{2n})(1+(e^{i\pi\tau})^{2n-1}(e^{iz})^2)(1+(e^{i\pi\tau})^{2n-1}(e^{iz})^{-2}). \tag{25}$$

We now consider the function

$$\vartheta_3((z-z_j+\omega)\sqrt{\frac{\pi}{2\mathfrak{D}}};i) = \sum_{m=-\infty}^{\infty} exp\left(i\pi i m^2 + 2mi(z-z_j+\omega)\sqrt{\frac{\pi}{2\mathfrak{D}}}\right), \tag{26}$$

$$= \sum_{m=-\infty}^{\infty}\left(e^{-\pi}\right)^{m^2}\left(e^{iz\sqrt{\frac{2\pi}{\mathfrak{D}}}}\right)^{2m}\left(e^{-i(z_j-\omega)\sqrt{\frac{\pi}{2\mathfrak{D}}}}\right)^{2m}. \tag{27}$$

Let

$$x = e^{-\pi}; y = exp\left[-i\left(\frac{2\pi}{\mathfrak{D}}\right)^{1/2}z\right]; b_j = (z_j-\omega)\left(\frac{\pi}{2\mathfrak{D}}\right)^{1/2}. \tag{28}$$

Then Eq. (26) is rewritten as

$$h(y) = \prod_{n=1}^{\infty}(1-x^{2n})(1+x^{2n-1}y^{-1}e^{-2b_ji})(1+x^{2n-1}ye^{2ib_j}). \tag{29}$$

We define

$$F(y) = \prod_{n=1}^{\infty}(1+x^{2n-1}ye^{2b_ji})(1+\frac{x^{2n-1}}{ye^{2ib_j}}), \tag{30}$$

$$= (1+xye^{2ib_j})(1+\frac{x}{ye^{2ib_j}})(1+x^3ye^{2ib_j})(1+\frac{x^3}{ye^{2ib_j}})(1+x^5ye^{2ib_j})(1+\frac{x^5}{ye^{2ib_j}})...$$

$$F(x^2y) = (1 + x^3ye^{2ib_j})(1 + \frac{1}{xye^{2ib_j}})(1 + x^5ye^{2ib_j})(1 + \frac{x}{ye^{2ib_j}})(1 + x^7ye^{2ib_j})(1 + \frac{x^3}{ye^{2ib_j}})...$$

$$\frac{F(x^2y)}{F(y)} = (1 + \frac{1}{xye^{2ib_j}})(\frac{1}{1 + xye^{2ib_j}}),$$

$$= \frac{xye^{2ib_j} + 1}{xye^{2ib_j}} \frac{1}{1 + xye^{2ib_j}},$$

$$= \frac{1}{xye^{2ib_j}},$$

$$F(y) = xye^{2ib_j} F(x^2y). \tag{31}$$

It is easily seen that

$$h(y) = F(y) \prod_{n=1}^{\infty} (1 - x^{2n}), \tag{32}$$

and

$$h(x^2y) = F(x^2y) \prod_{n=1}^{\infty} (1 - x^{2n}). \tag{33}$$

From Eq. (31), we see

$$h(x^2y) = \frac{F(y)}{xye^{2ib_j}} \prod_{n=1}^{\infty} (1 - x^{2n}) = \frac{h(y)}{xye^{2ib_j}}$$

$$h(y) = xye^{2ib_j} h(x^2y). \tag{34}$$

Then Eq. (21) is rewritten as

$$g(y) = cy^N \prod_{j=1}^{\mathfrak{D}} \prod_{n=1}^{\infty} (1 + x^{2n-1}y^{-1}e^{-2b_j i})(1 + x^{2n-1}ye^{2b_j i}), \tag{35}$$

$$= cy^N \prod_{j=1}^{\mathfrak{D}} h(y). \tag{36}$$

$$g(x^2y) = cx^{2N}y^N \prod_{j=1}^{\mathfrak{D}} h(x^2y), \tag{37}$$

from Eq. (34) we get

$$= cx^{2N}y^N \prod_{j=1}^{\mathfrak{D}} x^{-1}y^{-1}e^{-2ib_j} h(y), \tag{38}$$

$$= x^{-\mathfrak{D}}y^{-\mathfrak{D}}x^{2N} \prod_{j=1}^{\mathfrak{D}} e^{-2ib_j} g(y);$$

where

$$\prod_{j=1}^{\mathfrak{D}} e^{-2ib_j} = x^{-2N},$$

$$g(x^2y) = x^{-\mathfrak{D}}y^{-\mathfrak{D}}g(y). \tag{39}$$

Expand g in a Laurent Series:

$$g(y) = \sum_{n=-\infty}^{\infty} a_n y^n, \tag{40}$$

$$\sum_{n=-\infty}^{\infty} a_n y^n = x^{\mathfrak{D}}y^{\mathfrak{D}} \sum_{n=-\infty}^{\infty} a_n (x^2y)^n = \sum_{n=-\infty}^{\infty} a_n x^{2n+\mathfrak{D}} y^{n+\mathfrak{D}}, \tag{41}$$

This can be re-indexed with $n' = n - \mathfrak{D}$ on the left side of Eq. (41) to get

$$\sum_{n=-\infty}^{\infty} a_n y^n = \sum_{n=-\infty}^{\infty} a_n x^{2n-\mathfrak{D}} y^n, \tag{42}$$

$$a_n = a_{n-\mathfrak{D}} x^{2n-\mathfrak{D}}, \tag{43}$$

so

$$a_{\mathfrak{D}} = a_0 x^{\mathfrak{D}};$$
$$a_{\mathfrak{D}+1} = x^{2(\mathfrak{D}+1)-\mathfrak{D}} a_{\mathfrak{D}+1-\mathfrak{D}} = x^{\mathfrak{D}+2} a_1;$$
$$a_{2\mathfrak{D}} = x^{2(2\mathfrak{D})-\mathfrak{D}} a_{2\mathfrak{D}-\mathfrak{D}} = x^{3\mathfrak{D}} a_{\mathfrak{D}} = x^{3\mathfrak{D}}(a_0 x^{\mathfrak{D}}) = x^{4\mathfrak{D}} a_0;$$
$$a_{2\mathfrak{D}+1} = x^{2(2\mathfrak{D}+1)-\mathfrak{D}} a_{2\mathfrak{D}+1-\mathfrak{D}} = x^{3\mathfrak{D}+2-\mathfrak{D}} a_{\mathfrak{D}} + 1 = x^{3\mathfrak{D}+2-\mathfrak{D}}(x^{\mathfrak{D}+2} a_1) = x^{4\mathfrak{D}+\mathfrak{D}} a_1.$$

Therefore

$$a_{k\mathfrak{D}+j} = (x)^{(k^2\mathfrak{D}+2k_j)} a_j; \qquad j = 0, ..., \mathfrak{D} - 1. \tag{44}$$

On the other hand, we consider Eq. (5),

$$f(z) = \pi^{\frac{-1}{4}} \sum_{m=0}^{\mathfrak{D}-1} f_m \vartheta_3\left(\frac{\pi m}{\mathfrak{D}} - z\sqrt{\frac{\pi}{2\mathfrak{D}}}; \frac{i}{\mathfrak{D}}\right)$$

$$= \pi^{\frac{-1}{4}} \sum_{m=0}^{\mathfrak{D}-1} f_m \sum_{n=-\infty}^{\infty} (e^{-\pi})^{\frac{n^2}{\mathfrak{D}}} exp\left(\frac{inm\pi}{\mathfrak{D}}\right) exp\left(-i2nz\sqrt{\frac{\pi}{2\mathfrak{D}}}\right), \tag{45}$$

Let

$$x = exp(-\pi); \qquad y = exp\left(-i2z\sqrt{\frac{\pi}{2\mathfrak{D}}}\right)$$

Then Eq. (45) is rewritten as

$$g'(y) = \pi^{\frac{-1}{4}} \sum_{m=0}^{\mathfrak{D}-1} \sum_{n=-\infty}^{\infty} x^{\frac{n^2}{\mathfrak{D}}} y^n e^{(\frac{i2nm\pi}{\mathfrak{D}})} f_m, \tag{46}$$

$$= \pi^{\frac{-1}{4}} \sum_{n=-\infty}^{\infty} (x)^{n^2/\mathfrak{D}} y^n \sum_{m=0}^{\mathfrak{D}-1} f_m exp\left(\frac{2\pi imn}{\mathfrak{D}}\right), \tag{47}$$

$$= \pi^{\frac{-1}{4}} \sum_{n=-\infty}^{\infty} (x)^{n^2/\mathfrak{D}} y^n \tilde{f}_n, \tag{48}$$

$$= \sum_{n=-\infty}^{\infty} \left(\pi^{\frac{-1}{4}} (x)^{n^2/\mathfrak{D}} \tilde{f}_n\right) y^n, \tag{49}$$

If we let

$$a_n = \pi^{\frac{-1}{4}} (x)^{n^2/\mathfrak{D}} \tilde{f}_n, \tag{50}$$

Eq. (43) is satisfied,
and therefore $g(y) = g'(y)$. The equality of Eq. (5) and Eq. (21) is proved.

Presently we will get the values of F_m by Take advantage from the Eq. (20).
By substitution the values of roots in the equation

$$f(z_n) = \frac{1}{\sqrt{\sqrt{\pi}}} \sum_{m=0}^{\mathfrak{D}-1} F_m\, \theta[\frac{\pi m}{\mathfrak{D}} - z_n\sqrt{\frac{\pi}{\mathfrak{D}}}; \frac{i}{\mathfrak{D}}].o \tag{51}$$

Now we have a linear system of $\mathfrak{D}$ equations and $\mathfrak{D}$ variables,
by solving this system we get the coefficients.

4 Conclusion

This study presents an analysis of the importance of roots of the entire function.
These roots give a Sufficient definition for the limited dimension vector space.
By knowing the number of roots of the entire function we know the dimension of the limited dimension vector space.
The movement of these roots over the time has been studied.
The function has been built from its zeros.
A good and detailed proof has been provided.

Acknowledgments

We thank the Editor and the referee for their comments.

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